Dynamics of evolution Statistical physics Complex systems Stochastic processes

An interaction model on random networks

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Social movements, neurons in the brain or even industrial suppliers are best described by agents evolving on networks with basic interaction rules. In these real systems, the connectivity between agents corresponds to the a critical state of the system related to the noise of the system. The new idea is that connectivity adjusts itself because of two opposite tendencies: on the one hand information percolation is better when the network connectivity is small but all agents have rapidely the same state and the dynamics stops. On the other hand, when agents have a large connectivity, the state of a node (opinion of a person, state of a neuron, ...) tends to freeze: agents find always a minority among their neighbours to support their state. The model introduced here captures this essential feature showing a clear transition between the two tendencies at some critical connectivity. Depending on the noise, the dynamics of the system can only take place at a precise critical connectivity since, away from this critical point, the system remains in a static phase. When the noise is very small, the critical connectivity becomes very large, and highly connected networks are obtained like the airports network and the Internet. This model may be used as a starting point for understanding the evolution of agents living on networks.

What are the necessary conditions allowing a consensus among an assembly of voters? Why, sometimes, do all people share the same opinion in a short time? For example, we consider a school class of students who want to make an important travel, and have to choose between two destinations: Alaska or Rio de Janeiro. Every student has naturally a strong preference for one of the destinations. If students are isolated then nobody changes his opinion, and no consensus shall be found. When students take advice from one or two friends in the class, they may change their opinion quickly and a consensus will be achieved. Now if each student has many friends, he will always find a small group among his friends that shares his opinion and support his choice. Hence students will keep their opinion and no consensus is found although every student is connected to a large number of friends. In a neural network, the number of connections among neurons, dendritic tree, plays an important role like a social network. What is the critical number of incoming connections on a neuron? In the following article, we will see how to reproduce these behaviours using the language of opinion dynamics, and analyse the consequences on various fields like supply networks of firms or the airports networks.

There exist already several models where opinion dynamics of a community of agents is simulated. Among them are the voter model [1]: a 2-states spin is selected at random and it adopts the opinion of a randomly-chosen neighbour. This step is repeated until the system reaches consensus. Each agent has zero self confidence since he merely adopts the state of one of his neighbours. A similar model due to K. Sznajd-Weron and J. Sznajd [2] was designed to explain certain features of opinion dynamics resumed by the slogan "United we stand, divided we fal". This leads to a dynamics, in that individuals placed on a lattice can choose between two opinions, and in each update a pair of neighbours sharing common opinion persuade their neighbours to join their opinion. It is equivalent to the voter model as shown in ref. [3]. The local majority rule model described in ref. [4, 5] considers groups of agents where members adopt the opinion of the local majority.

It is commonly believed that correlated behaviour in financial markets or large opinion changes in human society is due to information that are shared by everyone at the same time. For example in a financial market, herding can be produced by the central market price, or by a rumor that propagates rapidly among traders. Information is collected from the action of each traders and is reflected in the price of stocks. However sometimes opinions changes occurs only because of local interaction among agents as we shall see later.

Some models describe the consequence of "herding" behaviour [6, 7, 8] in financial markets. However, in this class of models, herding behaviour is controlled by an external parameter. The question answered by this class of models is related to the consequence of herding on distribution of financial price data. In our case we are interested at finding the source of herding, why people follow sometimes the same trend in a short time although they seem to act independently.

The evolution of a genetic network has been described by S. Kauffman [9] using andom boolan networks (RNB). In RBN the state of a node is determined by the state of G neighbours depending a random lookup table. On the contrary, the model introduced here uses the same rule for all agents: the state of an agent changes only if all G neighbours have the opposite state. A temperature or noise allows agents to change their state by themselves. This rule is simpler, and maybe more general, compared to the RBN model since all the randomness is resumed in a single noise parameter r rather than in complex lookup tables.

The aim of the present work is to study the dynamics of a ensemble of agents connected on a random network, and to determine what are the conditions to have order or disorder under a fundamental interaction rule.

State - Each agent has a state $S \in \{-1, 0, +1\}$ with three possible values that can represent {vote A, do not vote, vote B} for a vote, or {buy, wait, sell} in a financial market. In real life, an agent can be a single person or a group of people that share the same opinion. The formation of states is determined by the confrontation, i.e. summation, of the state of the agent with the state of each neighbour (or advisor). Contrary to the voter model or the majority model [5], here agents have a the same self-confidence or strength as each of their neighbours, provided they have an state +1 or -1.

Algorithm (fixed connectivity G) - Now we consider a community of N agents where each agent i has a state $S_i(t) \in \{-1, 0, +1\}$ at time t. Each agent can be either a neighbour or being controlled by other agents. In each update, we sum the state of an agent picked at random with each state of G neighbours chosen at random among all agents. The sign of this sum represents the new state. More explicitly, at each time step t:

1. An agent i is selected at random.
2. The new state $S_i(t+1)$ is the sign of the sum of the state of the agent i with each state $S_{i_k}(t)$ of a random neighbours group $A_i = \{i_1, ..., i_G\}$:
$$S_i(t+1) = \text{sign}\left(\sum_{k=1}^{G}[S_i(t) + S_{i_k}(t)]\right) \quad (1)$$
where $\text{sign}(0) = 0$.
3. Instead of point 2., with probability r, the new state $S_i(t+1)$ is +1 or -1 taken at random.

The recursion relation (1) means that agents change their state only if all neighbours have an opposite state (unanimity rule). The algorithm is completely deterministic when no random state is introduced during the simulation, i.e. if $r = 0$. The situation where $r > 0$ is more realistic since real nodes of a system can change their state by some internal processus with the appearance of randomness for the neighbours.

The two-step change of state is realistic since nodes may have a state where they are not active, but this state is not essential to the dynamics of the system. A one-step change would essentialy lead to the same results.

The key parameter is the number of neighbours per agent or connectivity G. For $G = 1$ agent merely change the state when the neighbour has an opposite state. In this case the dynamics is similar, but no identical, to the voter model [1]. For $G = 2$, i.e. one agent and two neighbours, the algorithm is equivalent to the majority model [5]: if neighbours have the same state, they form the majority of the three agents (two neighbours against one agent). The two boundary cases are then:

Small connectivity (hierarchical society): each agent is controlled by a small number of neighbours and states can change easily. Starting from a random configuration, states or information scatter rapidly through the network resulting in a rapid ordering or consensus. A long range order appear due to strong correlations between neighbours and agents. The majority and voter models belong to this category, and are therefore opinion dynamics with weak self-confidence and rapid decision making.

Large connectivity (complex society): Agents tend to keep their own state because the probability that all neighbours have the same and opposite state is small. Information cannot be transmitted through the network, and eventually no long range order or consensus can emerge: states are essentially random. Interpretation: the connectivity G of an agent is proportional to his self-confidence: agents with many neighbours have large self-confidence since they keep their state when they find at least one neighbour sharing the same state.

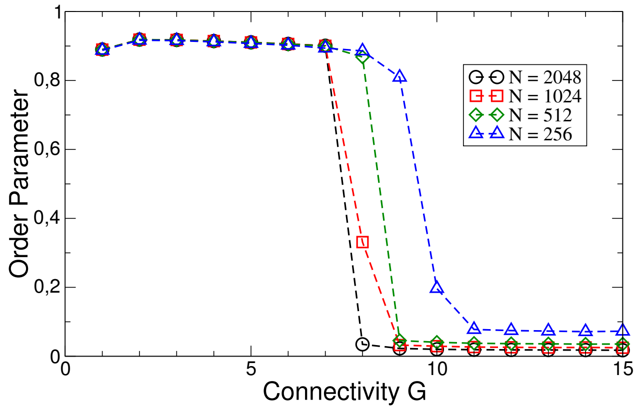

FIG. 1: Average absolute opinion (order parameter) $\tilde{S}(G)$ for connectivities $G \in \{1, 2, ...\}$ and different system sizes. The noise is $r = 0.05$. The transition between consensus and disorder is located near $G_c \approx 8$. The variance (not shown) is always 0 except for a non-zero order parameter at the transition (Here N=1024, G=8 and N=256, G=10).

Results - Computer simulations have been done using different values of the noise r and different group sizes. Statistics are done over 10 to 20 runs where each run has 10^4 updates per agent. As shown in figure 1 for $r = 0.05$, the order parameter

$$\tilde{S} := \frac{1}{N}\langle \, |\sum_i S_i(t)| \, \rangle_t, \quad (2)$$

which is averaged over time t, has a breakdown at a critical group size $G_c(r)$ marking a clear separation between two different regimes. For $G > G_c$, there is no global order. States are essentially random and $\tilde{S} = 0$. At $G = G_c$, states oscillate between ordered and random states. For $G << G_c$, states become correlated and rapidly evolve either to the average and finally $\tilde{S} = \pm 1$. A similar noise driven phase transition has been found for the majority model in ref. [4], and it corresponds in our model to the case $G = 2$.

When all links between agents are reciprocal no phase transition occurs: reciprocity reduce the transmission of information through the network.

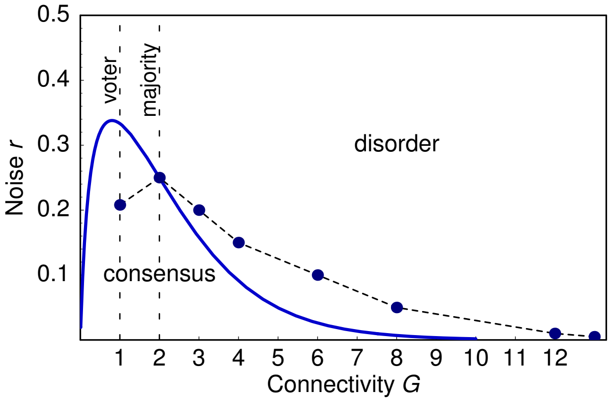

FIG. 2: Phase diagram in the $\{r, G\}$ plane showing two different regimes: consensus and disorder. The thick line is the analytical solution $r_c(G)$ from equation (7). Points are results from computer simulations.

Analytical Approach for the static model - The critical point G_c, found in simulations, separating the correlated phase and the random phase can be derived analytically. To do that, we consider a system with an infinite number of agents, and we neglect effects of loops. The connectivity G of an agent is then equal in average to the number of agents that an agent advises, i.e. $G = G_{out}$.

Now we look for the probability P that an agent changes his state. Only states that are -1 or +1 are taken into account since 0 states disappear quickly. If we have n agents with state +1, then $x = n/N$ is the probability of finding an agent with state +1. If the noise is zero, the probability P_G that an agent i changes his state from -1 to +1 or from +1 to -1 is given by the probability to find the agent i with state -1 and G neighbours with states +1 plus the corresponding probability to find the agent i with state +1 and G neighbours with states -1:

$$P_G = \frac{G}{1+G}\ \left[x(1-x)^G + (1-x)x^G\right] \qquad (3)$$

where the factor $\frac{G}{1+G}$ is introduced in order to take into account that empty groups, i.e. $G = 0$, induce no state change. If we add a noise r with uniform distribution between 0 and 1, the total probability P of changing the state is 1/2 with probability r plus P_G with probability $1-r$:

$$P = \frac{r}{2} + (1-r)P_G. \qquad (4)$$

Consensus is reached when the probability of changing the state because of the neighbours is larger than the probability of changing randomly the state:

$$(1-r)P_G > r/2 \quad \Rightarrow \quad \text{consensus}. \qquad (5)$$

For a random configuration with $x = 1/2$, the critical noise r_c, which separates consensus and disorder, is determined by the condition $(1-r_c)P_G = r_c/2$. This leads to:

$$(1-r_c)\frac{G}{1+G}\frac{1}{2^G} = \frac{r_c}{2} \qquad (6)$$

Solving the last equation with respect to r_c gives the formula:

$$r_c = \frac{1}{1 + 2^{G-1}\left(1+G^{-1}\right)}. \qquad (7)$$

r_c has a maximum at $G \approx 0.801$.

In figure 2, the results of simulations are shown together with the analytical results from equation (7) showing a qualitative good agreement. Note that the majority rule ($G = 2$) leads to a larger consensus phase than $G = 1$ (dynamics similar to the voter model). The transition points for the infinite system are obtained by computing the intersection of reduced fourth order cumulants [10] for different system sizes.

Dynamical Groups - Until now, each agent had the same number of neighbours. A drawback of this static approach is that the number of neighbours is a discrete quantity. Hence it is not possible to study the phase transition as a function of a continuous parameter. Moreover a fixed number of neighbours is not very realistic because agents usually have different numbers of neighbours. In order to get closer to reality and to study the transition with a continuous parameter, neighbours groups are now formed according to a probability p of growing the group size by one neighbour. The algorithm starts with empty groups, and at each time step t:

1. An agent i is selected at random.
2. With probability p, increase the group size by one neighbour: $G_i \rightarrow G_i+1$. Otherwise, i.e. with probability $1-p$, perform point 2. of the static model, and remove one neighbour: $G_i \rightarrow G_i - 1$.
3. Instead of point 2., with probability r, a new state +1 or -1 is taken at random.

In figure 3, the order parameter $\tilde{S}$ is plotted for different values of the noise r for 10^4 steps and 20 runs. Following a line defined by a constant order parameter: when p is very small, most agents have zero neighbour, and information cannot percolate through the network ($\tilde{S} = 0$). When p increases, there is a critical probability p where agents have enough connections in order to establish consensus in the entire network, and $\tilde{S} > 0$. If we increase again p, a second transition occurs where groups of neighbours are so large that agents cannot change their state anymore ($\tilde{S} = 0$). Simulations for $r = 0$ are not conclusive since they suffer of a very slow dynamics. The standard deviation of $\tilde{S}$ is also reported and it exhibits maxima at the two transition points.

The phase diagram of the dynamical model is shown in figure 4. The transition points for the infinite system are obtained with fourth order cumulants for different system

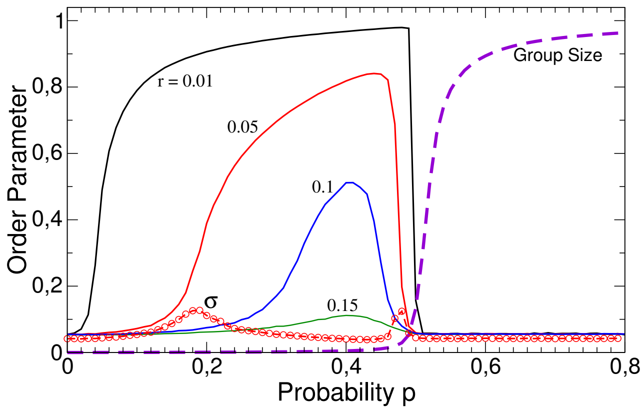

FIG. 3: Order parameter $\tilde{S}$ of the dynamical model for $N = 400$ agents and different value of noise r, and the corresponding average relative group size $\langle G_i \rangle / N$. The circles show the standard deviation σ of $\tilde{S}$ for $r = 0.05$.

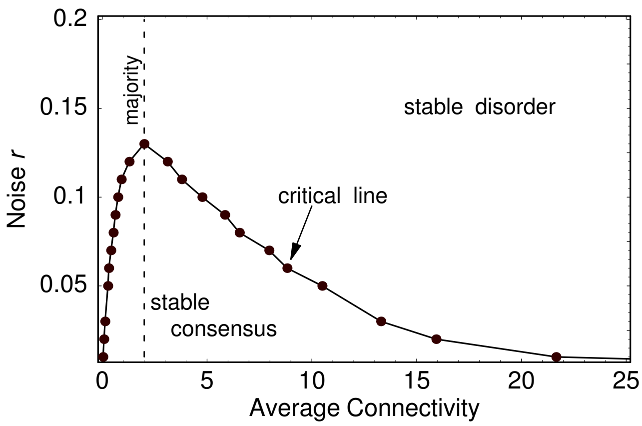

FIG. 4: Phase diagram - the critical connectivity (points) marks the transition between stable consensus and stable disordered phases. The transition coincides with the maximum of the standard deviation or activity σ of the order parameter. Computer simulations have been done for different noise r and probability p.

sizes. We note that for small groups G, the transition occurs at a smaller noise r compared with the static model shown in figure 2 whereas r_c is larger for big groups.

An interesting feature is that the majority rule $G = 2$ is the less noise sensitive of all decision procedures. This is a hint showing that communities use in general the majority rule because it is the most error tolerant system.

A necessary condition of the evolution - we consider now the class of systems evolving on random network and constrained by the unanimity rule. These systems can be a social network, industrial supply networks or airports networks. Considering a given error tolerance or noise, there is a unique average critical connectivity $\langle G_c \rangle$ where the system has a phase transition between a consensus phase and a disordered phase. If the system decreases its connectivity, individuals or nodes can change their state according to their neighbours resulting in a rapid consensus: then all nodes have the same state. This consensus phase is stable and does not allow nodes to change their state anymore and no evolution is possible. When the system increases its connectivity, nodes tend to keep their states ending in a disordered phase. This disordered phase remains frozen as long as the connectivity is large and nodes keep their state. In this static phase, the system cannot move from one configuration to another, and therefore no evolution can take place. The only point where this class of systems can change their state is at the average critical connectivity $\langle G_c \rangle$. Depending on the particular noise of the system, at $\langle G_c \rangle$, the agents of the systems change all the time between consensus and disorder. Since the system is in a critical and unstable state, a small perturbation in one node can then result in avalanche of changes in a large part of the system. This critical state is therefore the only region where the evolution of the system can take place. Note that the evolution of the underlying random network itself is a related but different problem. This simple model is not a complete model of evolution because there is no selection rules or genetic evolution like in real systems.

Social networks - In a real networks of people, the probability p can be interpreted as a "social pressure" which forces people to be near an ideal number of advisors (neighbours). Extreme opinions that propagate rapidly are dangerous for the stability of the social cohesion. On the other side, people taking into account too many different sources of information are unable to change their opinion. Hence there is a critical number of neighbours $\langle G_c \rangle$ that allows the system to change its state. If we define the "social activity" as the variance σ of the order parameter $\tilde{S}$, then σ is maximum precisely at the transition as shown in figure 3. We consider only the upper transition, and not the percolation transition where G_c is small. This criticality is related to the concept self-organised criticality as introduced by P. Bak *et al* in ref. [11]. In a real process of decision making, the main difficulty is to estimate the noise level that is present. For example a noise level of $r = 0.1$, which seems reasonable, leads to $G_c \approx 5$. Of course the number of real contacts, either groups or single persons, of an agent is larger than G_c since not every contact is an advisor. The random network of the present model can be therefore different from the physical network since advisors may consist of group of persons (for example, the family).

Supply networks - A complex object like a car needs several components produced by other firms. These firms transform material produced by other firms as well. Hence the flux of products forms a network where the product can successfully reach its destination (+1) or not (-1). Like in other networks, there is a critical connectivity in the firms network: it is more difficult to have hundred of suppliers since coordination costs increases with the coordination number. On the other hand, having a few suppliers induces dependence and decreases flexibility. Therefore for each suppliers network, there is an critical number G_c of suppliers depending on the error tolerance r. These conclusions can be seen as a generalisation of empirical cost studies for a single firm as a function of the number of suppliers (see [12]).

Networks with large connectivity - a special case is obtained for zero noise, $r \rightarrow 0$: the critical connectivity tends to infinity, i.e. $G_c \rightarrow \infty$ as shown by equation (7). The network never finds its equilibrium and, although a finite average connectivity can be calculated or measured, it does not correspond to the critical connectivity G_c that goes to infinity. In the airports network, the noise level is very low ($r \rightarrow 0$): airports are only in function when all airplanes arrive at destination (unanimity rule). The routers in the Internet network are subject to zero error tolerance as well. A router is operating only when all incoming packets are distributed without error. Hence, nodes of these networks can accumulate connections as long as the noise r is very low, and they become very large networks.

In conclusion, a model is introdued where agents with possible states -1,0 or 1 interact with G neighbourgs picked at random. The single rule is that an agent changes the state only if all neighbourgs have the opposite state. Appart from the rule, agents can spontanously change their state at random. When varying the connectivity G, a phase transition occurs at a certain value G_c between a phase of consensus and a phase of disorder. G_c depends on the noise or error tolerance of the system and on the particular underlying network. When agents have a few number of neighbourgs, information percolates through the network since agents change their opinion frequently, and a consensus is found. In the case of large advisor groups, agents tend to keep their state since large groups have less frequently the opposite state as the agent, and a little noise causes disorder. By varying the connectivity, one can tune the states of agents. This model is a generalisation of existing opinion models: for one neighbour, the dynamics is similar to the voter model, and for two neighbours it is equivalent to a majority rule. Finally, the idea is put forward that real social networks and industrial networks organise themselves at the maximum of activity, precisely at the critical connectivity G_c that sets a scale for the system. The airports network or the Internet, where error tolerance is very low, have a large critical connectivity tending to infinity.

A future issue is the role of the structure of the network and the possibility to have heterogenous noise in the system. Another issue is to measure or implement the complexity of agents.

[1] P. Clifford and A. Sudbury, Biometrika **60**, 581 (1973).
[2] K. Sznajd-Weron and J. Sznajd, International Journal of Modern Physics C **11**, 1157 (2000).
[3] L. Behera and F. Schweitzer, International Journal of Modern Physics C **14**, 1331 (2003).
[4] C. Huepe-Minoletti and M. Aldana-Gonzalez, Journal of Statistical Physics **108**, 527 (2002).
[5] P. L. Krapivsky and S. Redner, Phys. Rev. Lett. **90**, 238701 (2003).
[6] R. Cont and J. Bouchaud, Macroeconomic Dynamics **4**, 170 (2000).
[7] V. Eguíluz and M. Zimmermann, Phys. Rev. Lett. **85**, 5659 (2003).
[8] Ph. Curty and M. Marsili, J. Stat. Mech., P03013 (2006).
[9] S. Kauffman *The Origins of Order*, Oxford University Press, (1993)
[10] K. Binder, Z. Phys. B **43**, 119 (1981).
[11] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[12] J. Y. Bakos and E. Brynjolfsson, Journal of Organizational Computing **3**, 3 (1993).
[13] S. Bornholdt and T. Roehl, Phys. Rev. E **67**, 066118 (2003).